INDECOMPOSABILITY OF FREE GROUP FACTORS OVER NONPRIME SUBFACTORS AND ABELIAN SUBALGEBRAS

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ABSTRACT. We use the free entropy defined by D. Voiculescu to prove that the free group factors can not be decomposed as closed linear spans of noncommutative monomials in elements of nonprime subfactors or abelian $*$-subalgebras, if the degrees of monomials have an upper bound depending on the number of generators. The resulting estimates for the hyperfinite and abelian dimensions of free group factors settle in the affirmative a conjecture of L. Ge and S. Popa (for infinitely many generators).

1. INTRODUCTION

L. Ge and S. Popa defined ([GePo]) for a given type II_1-factor $\mathcal{M}$ the following two quantities: $\ell_h(\mathcal{M}) = \min\{f \in \mathbb{N} \mid \exists$ hyperfinite $\mathcal{R}_1, \ldots, \mathcal{R}_f \subset \mathcal{M}$ such that $\overline{\mathrm{sp}}^w \mathcal{R}_1\mathcal{R}_2 \ldots \mathcal{R}_f = \mathcal{M}\}$, $\ell_a(\mathcal{M}) = \min\{f \in \mathbb{N} \mid \exists$ abelian $\mathcal{A}_1, \ldots, \mathcal{A}_f \subset \mathcal{M}$ such that $\overline{\mathrm{sp}}^w \mathcal{A}_1\mathcal{A}_2 \ldots \mathcal{A}_f = \mathcal{M}\}$ (the min considered is ∞ if $\mathcal{M}$ can not be generated as stated) and conjectured that $\ell_h(\mathcal{L}(\mathbb{F}_n)) = \ell_a(\mathcal{L}(\mathbb{F}_n)) = \infty$ for $n \geq 2$, where $\mathcal{L}(\mathbb{F}_n)$ is the type II_1-factor associated to the free group with n generators.

We use the concept of free entropy introduced by D. Voiculescu in his breakthrough paper [Vo2] to prove that the conjecture mentioned above is true at least partially (for $n = \infty$) that is, $\ell_h(\mathcal{L}(\mathbb{F}_n)), \ell_a(\mathcal{L}(\mathbb{F}_n)) \geq [\frac{n-2}{2}] + 1$ for all $4 \leq n \leq \infty$. Actually, our result is more general and it states that the free group factor with n generators can not be asymptotically generated (Definitions 3.2 and 4.2) as

$$\lim_{\omega \to 0}{}^{\|\cdot\|_2} \sum_{\substack{1 \leq j_1, \ldots, j_{t+1} \leq f \\ 1 \leq t \leq d}} \mathcal{N}^\omega_{j_1} \mathcal{Z}^\omega \mathcal{N}^\omega_{j_2} \mathcal{Z}^\omega \ldots \mathcal{N}^\omega_{j_t} \mathcal{Z}^\omega \mathcal{N}^\omega_{j_{t+1}}$$

or

$$\lim_{\omega \to 0}{}^{\|\cdot\|_2} \sum_{\substack{1 \leq j_1, \ldots, j_{t+1} \leq f \\ 1 \leq t \leq d}} \mathcal{A}^\omega_{j_1} \mathcal{Z}^\omega \mathcal{A}^\omega_{j_2} \mathcal{Z}^\omega \ldots \mathcal{A}^\omega_{j_t} \mathcal{Z}^\omega \mathcal{A}^\omega_{j_{t+1}}$$

if $\{\mathcal{N}^\omega_1, \ldots, \mathcal{N}^\omega_f\}_\omega$ are nonprime subfactors, $\{\mathcal{A}^\omega_1, \ldots, \mathcal{A}^\omega_f\}_\omega$ are abelian $*$-subalgebras, $\{\mathcal{Z}^\omega \subset \mathcal{L}(\mathbb{F}_n)\}_\omega$ are subsets containing p self-adjoint elements, and $f, d \geq 1$ are integers such that $n \geq p + 2f + 1$. Note that $\mathcal{L}(\mathbb{F}_n)$

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admits decompositions of this sort if we allow $d = \infty$, for example if $\mathcal{Z}^\omega = \mathcal{Z} = \{1\}, f = n$, $\mathcal{N}_1^\omega = \mathcal{N}_1, \dots, \mathcal{N}_n^\omega = \mathcal{N}_n$ are n distinct copies of the hyperfinite type II_1-factor $\mathcal{R}$ and $\mathcal{A}_1^\omega = \mathcal{A}_1, \dots, \mathcal{A}_n^\omega = \mathcal{A}_n$ are n distinct copies of $L^\infty([0,1])$ (since $\mathcal{L}(\mathbb{F}_n)$ is both the free product of n copies of $\mathcal{R}$ and the free product of n copies of $L^\infty([0,1])$, see [VDN]). Note also that the indecomposability of $\mathcal{L}(\mathbb{F}_n)$ as $\overline{\mathrm{sp}}^w \mathcal{N}\mathcal{Z}\mathcal{N}$ implies the primeness of its subfactors ([Şt]). Indeed, according to V. Jones ([Jo]), if $\mathcal{N}$ is a subfactor of finite index in $\mathcal{M}$ then $\mathcal{M}$ decomposes as $\mathcal{N}e\mathcal{N}$ where e is the Jones projection. In particular, the indecomposability properties of $\mathcal{L}(\mathbb{F}_n)$ over nonprime subfactors and abelian subalgebras are preserved to its subfactors of finite index. Recall that the Haagerup approximation property ([Ha]) is another property preserved to the free group subfactors. A first example of a prime II_1-factor (with a nonseparable predual, though) was given by S. Popa ([Po1]) and then L. Ge proved (with a free entropy estimate) that the free group factor $\mathcal{L}(\mathbb{F}_n)$ is prime $\forall 2 \le n < \infty$ ([Ge2]), thus answering a question from [Po3].

Our results are based on estimates of free entropy that is, estimates of volumes of various sets of matrix approximants (matricial microstates). The paper has four parts. After introduction, we prove the first estimate of free entropy and reobtain then a result of D. Voiculescu ([Vo2]): if a free family of m self-adjoint noncommutative random variables can be generated by noncommutative power series by another family of n self-adjoint noncommutative random variables, then $n \ge m$ (Theorem 2.3). However, we show that the assumption of freeness from [Vo2] is not essential and it can be dropped. As a consequence, the number of self-adjoint generators with finite entropy, which generate a $*$-algebra $\mathcal{A}$ *algebraically*, is constant. In the third part we prove the indecomposability of $\mathcal{L}(\mathbb{F}_n)$ (and of its subfactors of finite index) over nonprime subfactors (Theorem 3.5) and in the last section, the indecomposability over abelian subalgebras (Theorem 4.4).

We give next a short account on Voiculescu's free probability theory ([Vo1], [VDN]) and on his original concept of free entropy ([Vo2], [Vo3]). A type II_1-factor $\mathcal{M}$ endowed with its unique normalized, faithful, normal trace τ is sometimes called a W^*-probability space. The trace τ determines the 2-norm on $\mathcal{M}$, $||x||_2 = \tau(x^*x)^{\frac{1}{2}}$ and the completion of $\mathcal{M}$ w.r.t. $||\cdot||_2$ is denoted $L^2(\mathcal{M}, \tau)$. An element $x \in \mathcal{M}$ is a semicircular element if it is self-adjoint and if its distribution is given by the semicircle law:

$$\tau(x^k) = \frac{2}{\pi}\int_{-1}^{1} t^k \sqrt{1-t^2}\,dt \ \forall k \in \mathbb{N}.$$

A family $(\mathcal{A}_i)_{i\in I}$ of unital $*$-subalgebras of $\mathcal{M}$ is a free family provided that $\tau(x_1 x_2 \dots x_n) = 0$ whenever $\tau(x_k) = 0$, $x_k \in \mathcal{A}_{i_k}$ $\forall 1 \le k \le n,\ i_1, \dots, i_n \in I$ and $i_1 \ne i_2 \ne \dots \ne i_n$, $n \in \mathbb{N}$. A set $\{x_i\}_{i\in I} \subset \mathcal{M}$ is free if the family $(*\text{-alg}\{1, x_i\})_{i\in I}$ is free. A free set $\{x_i\}_{i\in I} \subset \mathcal{M}$ consisting of semicircular elements is called a semicircular system. If $\mathbb{F}_n$ is the free group with n generators ($2 \le n \le \infty$) then $\mathcal{L}(\mathbb{F}_n)$ denotes ([MvN]) the von Neumann algebra

generated by the left regular representation $\lambda : \mathbb{F}_n \to \mathcal{B}(l^2(\mathbb{F}_n))$. $\mathcal{L}(\mathbb{F}_n)$ is a factor of type II_1 - the free group factor on n generators. It has a canonical trace $\tau(\cdot) = (\cdot\delta_e, \delta_e)$, where $\{\delta_g\}_{g\in\mathbb{F}_n}$ is the standard orthonormal basis in $l^2(\mathbb{F}_n)$. Every $\mathcal{L}(\mathbb{F}_n)$ is generated as a von Neumann algebra by a semicircular system with n elements ([VDN]). We denote by $\mathcal{M}_k^{sa} = \mathcal{M}_k^{sa}(\mathbb{C})$ the set of $k \times k$ self-adjoint complex matrices and by τ_k its unique normalized trace. τ_k induces the 2-norm $||\cdot||_2 : \mathcal{M}_k^{sa} \to \mathbb{R}_+$ and the euclidean norm $||\cdot||_e := \sqrt{k}||\cdot||_2$. If B is a measurable subset of a m dimensional (real) manifold then $\mathrm{vol}_m(B)$ will denote the Lebesgue measure of B. The free entropy $\chi(x_1, \ldots, x_n)$ of a finite family of self-adjoint elements was introduced in [Vo2] but we will recall ([Vo3]) the definition of the modified free entropy which is better suited for applications. For self-adjoint elements $x_1, \ldots, x_{n+m} \in \mathcal{M}$ one defines first the set of matricial microstates

$$\begin{aligned}(1)\qquad &\Gamma_R(x_1, \ldots, x_n : x_{n+1}, \ldots, x_{n+m}; p, k, \epsilon) := \{(A_1, \ldots, A_n) \in (\mathcal{M}_k^{sa})^n | \\ &\quad \exists (A_{n+1}, \ldots, A_{n+m}) \in (\mathcal{M}_k^{sa})^m \text{ s.t. } ||A_j|| \le R,\ \big|\tau(x_{i_1} \ldots x_{i_q}) \\ &\quad -\tau_k(A_{i_1} \ldots A_{i_q})\big| < \epsilon\ \forall j, i_1, \ldots i_q\ \in \{1, \ldots, n+m\}\ \forall 1 \le q \le p\}\end{aligned}$$

where $R, \epsilon > 0$ and $p, k \in \mathbb{N}$, and then

$$\begin{aligned}(2)\qquad &\chi_R(x_1, \ldots, x_n : x_{n+1}, \ldots, x_{n+m}; p, k, \epsilon) \\ &\quad = \log(\mathrm{vol}_{nk^2}(\Gamma_R(x_1, \ldots, x_n : x_{n+1}, \ldots, x_{n+m}; p, k, \epsilon))),\end{aligned}$$

$$\begin{aligned}(3)\qquad &\chi_R(x_1, \ldots, x_n : x_{n+1}, \ldots, x_{n+m}; p, \epsilon) \\ &\quad = \limsup_{k\to\infty} \left(\frac{1}{k^2} \chi_R(x_1, \ldots, x_n : x_{n+1}, \ldots, x_{n+m}; p, k, \epsilon) + \frac{n}{2} \log k \right),\end{aligned}$$

$$\begin{aligned}(4)\qquad &\chi_R(x_1, \ldots, x_n : x_{n+1}, \ldots, x_{n+m}) \\ &\quad = \inf\{\chi_R(x_1, \ldots, x_n : x_{n+1}, \ldots, x_{n+m}; p, \epsilon) | p \in \mathbb{N}, \epsilon > 0\},\end{aligned}$$

$$\begin{aligned}(5)\qquad &\chi(x_1, \ldots, x_n : x_{n+1}, \ldots, x_{n+m}) \\ &\quad = \sup\{\chi_R(x_1, \ldots, x_n : x_{n+1}, \ldots, x_{n+m}) | R > 0\}.\end{aligned}$$

When taking the last sup it suffices though to assume $0 < R \le \max\{||x_1||, \ldots, ||x_{n+m}||\}$ rather than $0 < R < \infty$ ([Vo2], [Vo3]). The quantity $\chi(x_1, \ldots, x_n : x_{n+1}, \ldots, x_{n+m})$ is the free entropy of $x_1, \ldots, x_n$ in the presence of $x_{n+1}, \ldots, x_{n+m}$. If $m = 0$ it is called the free entropy of $x_1, \ldots, x_n$ and denoted $\chi(x_1, \ldots, x_n)$. If $\{x_{n+1}, \ldots, x_{n+m}\} \subset \{x_1, \ldots, x_n\}''$ then ([Vo3])

$$\chi(x_1, \ldots, x_n : x_{n+1}, \ldots, x_{n+m}) = \chi(x_1, \ldots, x_n).$$

For a single self-adjoint element $x = x^* \in \mathcal{M}$ one has ([Vo2]):

$$\chi(x) = \frac{3}{4} + \frac{1}{2} \log 2\pi + \int\int \log|s - t| d\mu(s) d\mu(t),$$

where μ is the distribution of x. If $x_1, \ldots, x_n$ are n self-adjoint free elements of $\mathcal{M}$ then $\chi(x_1, \ldots, x_n) = \chi(x_1) + \ldots + \chi(x_n)$ ([Vo2]). The converse is also true ([Vo4]), provided that $\chi(x_i) > -\infty\ \forall 1 \le i \le n$. In particular, the free

entropy of a finite semicircular system is finite, hence the free group factor $\mathcal{L}(\mathbb{F}_n)$ has a system of generators with finite free entropy for $2 \leq n < \infty$.

2. Noncommutative power series and free entropy

The main result of the present section states that if a (not necessarily free) family of m self-adjoint noncommutative random variables with finite free entropy can be generated as noncommutative power series by another family of n self-adjoint noncommutative random variables, then $n \geq m$. In other words, a finite system with finite free entropy has minimal cardinality among all finite systems of self-adjoint elements that are equivalent under the noncommutative analytic functional calculus. Thus, we recover D. Voiculescu's result from [Vo2], with the observation that our approach does not require the assumption of freeness.

We review first a few facts concerning the theory of systems of algebraic equations ([vdW]), necessary in the proof of Lemma 2.1. If $g_1, \ldots, g_n$ are forms in n variables, then there exists a polynomial (the rezolvent) in their coefficients, $R(g_1, \ldots, g_n)$, with the property that $R(g_1, \ldots, g_n) = 0$ if and only if the system $g_1(\xi_1, \ldots, \xi_n) = \ldots = g_n(\xi_1, \ldots, \xi_n) = 0$ has a nontrivial solution. If $h_1, \ldots, h_{n-1}$ are $n-1$ forms in n variables and $h_n(u)(\xi_1, \ldots, \xi_n) := u_1\xi_1 + \ldots + u_n\xi_n$, then $R_u(h_1, \ldots, h_{n-1}) := R(h_1, \ldots, h_{n-1}, h_n(u))$ (the u-rezolvent) is either identically equal to 0, or a form of degree $\deg(h_1) \cdot \ldots \cdot \deg(h_{n-1})$ in $u = (u_1, \ldots, u_n)$. In the first case, the system $h_1 = \ldots = h_{n-1} = 0$ has infinitely many solutions $[(\xi_1, \ldots, \xi_n)] \in \mathbb{PC}^{n-1}$ and in the second, all the solutions $[(\xi_1, \ldots, \xi_n)] \in \mathbb{PC}^{n-1}$ are given by the factorization of $R_u(h_1, \ldots, h_{n-1})$ (and thus, the system admits at most $\deg(h_1) \cdot \ldots \cdot \deg(h_{n-1})$ solutions - Bézout's Theorem).

Let $f_1, \ldots, f_n \in \mathbb{R}[\Xi_1, \ldots, \Xi_n]$ be n polynomials in n indeterminates, of degrees $d_1, \ldots, d_n$, respectively. For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ define

$$F_{i,a_i}(\xi_1, \ldots, \xi_{n+1}) = \xi_{n+1}^{d_i}\left(f_i\left(\frac{\xi_1}{\xi_{n+1}}, \ldots, \frac{\xi_n}{\xi_{n+1}}\right) - a_i\right), \forall 1 \leq i \leq n.$$

Bézout's Theorem implies that the system of equations $f_1(\xi_1, \ldots, \xi_n) = a_1, \ldots, f_n(\xi_1, \ldots, \xi_n) = a_n$ admits at most $d_1 \cdot \ldots \cdot d_n$ solutions $(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ if $R_u(F_{1,a_1}, \ldots, F_{n,a_n}) \not\equiv 0$. Note also that the set

$$S_u(f_1, \ldots, f_n) := \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid R_u(F_{1,a_1}, \ldots, F_{n,a_n}) \not\equiv 0\}$$

is either open and dense in $\mathbb{R}^n$, or empty.

We proceed now with Lemma 2.1 which gives an upper bound for the Lebesgue measure of the intersection of an algebraically parameterized manifold embedded in $\mathbb{R}^m$, with the unit ball of $\mathbb{R}^m$. This Lemma will be of further use in estimating the volumes of various sets of matricial microstates which will appear as sets of points within given distance from such manifolds.

Lemma 2.1. *For integers $n \leq m$ and polynomials $f_1, \ldots, f_m \in \mathbb{R}[\Xi_1, \ldots, \Xi_n]$ define $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$. If the polynomials* $\det\left(\frac{\partial f_J}{\partial \xi}\right)$

are not identically equal to 0 $\forall J \in \{(i_1,\dots,i_n)|1 \le i_1 < \dots < i_n \le m\}$ *and if* $S_u = S_u(f_1,\dots,f_n) \neq \emptyset$*, then*

$$\int_{f^{-1}\left(\overline{B(0,1)}\right)} \left(\sum_{|J|=n} \det{}^2 \left(\frac{\partial f_J}{\partial \xi} \right) \right)^{\frac{1}{2}} d\xi \le \binom{m}{n} \cdot C \cdot vol_n(B(0,1)) \,, \tag{1}$$

where $C = C(\deg(f)) = \max\{\deg(f_{i_1})\cdot\ldots\cdot\deg(f_{i_n}) \mid 1 \le i_1 < \dots < i_n \le m\}$ *and* $B(0,1) = B_n(0,1)$ *is the unit ball in* $\mathbb{R}^n$*.*

Proof. We consider first the case $m = n$. Let S denote the set of all irregular values of f, $S = f\left(\{\xi \in \mathbb{R}^n \mid \operatorname{rank}(df_\xi) < n\}\right)$. It suffices to show that (1) holds with $f^{-1}\left(\overline{B(0,1)} \setminus S_\epsilon\right)$ replacing $f^{-1}\left(\overline{B(0,1)}\right)$, where S_ϵ is an arbitrary open set that contains $S \cup (\mathbb{R}^n \setminus S_u)$. For any $a = (a_1,\dots,a_n) \in \operatorname{Range}(f) \cap \overline{B(0,1)} \setminus S_\epsilon$ the set $f^{-1}(\{a\})$ has at most $C = \deg(f_1)\cdot\ldots\cdot\deg(f_n)$ elements, say $f^{-1}(\{a\}) = \{b_1,\dots,b_{p(a)}\}$ for some $1 \le p(a) \le C$. There exist an open ball $B_a \ni a$ and open neighborhoods $V_1^a \ni b_1,\dots,V_{p(a)}^a \ni b_{p(a)}$ such that B_a and V_i^a are diffeomorphic via f for $1 \le i \le p(a)$ and $f^{-1}(B_a) = \cup_{i=1}^{p(a)} V_i^a$. Since it is compact, we can cover $\operatorname{Range}(f) \cap \overline{B(0,1)} \setminus S_\epsilon$ with a finite set of such open balls $B_{a_1},\dots,B_{a_k}$. This covering determines a finite partition of $\operatorname{Range}(f) \cap \overline{B(0,1)} \setminus S_\epsilon$, say $W_1,\dots,W_t$. For each $1 \le j \le t$ choose a unique $1 \le l = l(j) \le k$ such that $W_j \subset B_{a_l}$ and $f^{-1}(W_j) = T_{j1} \cup \ldots \cup T_{jp(a_l)}$ where $T_{ji} \subset V_i^{a_l}$ and W_j and T_{ji} are diffeomorphic via f for all $1 \le i \le p(a_l)$.

$$\begin{aligned} \int_{f^{-1}\left(\overline{B(0,1)} \setminus S_\epsilon\right)} \left| \det\left(\frac{\partial f}{\partial \xi}\right) \right| d\xi &= \sum_{j=1}^{t} \int_{f^{-1}(W_j)} \left| \det\left(\frac{\partial f}{\partial \xi}\right) \right| d\xi \\ &= \sum_{j=1}^{t} \sum_{i=1}^{p(a_{l(j)})} \int_{T_{ji}} \left| \det\left(\frac{\partial f}{\partial \xi}\right) \right| d\xi = \sum_{j=1}^{t} \sum_{i=1}^{p(a_{l(j)})} \operatorname{vol}_n(W_j) \\ &\le C \sum_{j=1}^{t} \operatorname{vol}_n(W_j) = C \cdot \operatorname{vol}_n\left(\overline{B(0,1)} \setminus S_\epsilon\right). \end{aligned} \tag{2}$$

In the case $m > n$ one has the following estimates:

$$\begin{aligned} \int_{f^{-1}\left(\overline{B(0,1)}\right)} \left(\sum_{|J|=n} \det{}^2 \left(\frac{\partial f_J}{\partial \xi} \right) \right)^{\frac{1}{2}} d\xi &\le \int_{f^{-1}\left(\overline{B(0,1)}\right)} \sum_{|J|=n} \left| \det\left(\frac{\partial f_J}{\partial \xi}\right) \right| d\xi \\ \le \sum_{|J|=n} \int_{f_J^{-1}\left(\overline{B(0,1)}\right)} \left| \det\left(\frac{\partial f_J}{\partial \xi}\right) \right| d\xi &\le \binom{m}{n} \cdot C \cdot \operatorname{vol}_n(B(0,1)). \end{aligned}$$

$\square$

Lemma 2.1 will be used in the proof of Proposition 2.2. The $k \times k$ matricial microstates of $x_1,\dots,x_m$ are points within euclidean distance $2\omega\sqrt{mk}$ from the range of a polynomial function in the matricial microstates of $y_1,\dots,y_n$

provided that each x_i is within $||\cdot||_2$-distance ω from noncommutative polynomials in $y_1, \ldots, y_n$.

Proposition 2.2. *Let $P_1, \ldots, P_m \in \mathbb{C} < Y_1, \ldots, Y_n >$ be complex polynomials in n noncommutative self-adjoint variables. Assume that $(\mathcal{M}, \tau)$ is a II_1-factor and $\{x_1, \ldots, x_m\} \subset \mathcal{M}$ is a finite set of self-adjoint generators of $\mathcal{M}$. If $\{y_1, \ldots, y_n\} \subset \mathcal{M}$ is another finite set of self-adjoint generators of $\mathcal{M}$ with $n < m$ and such that*

$$||x_i - P_i(y_1, \ldots, y_n)||_2 < \omega \ \forall\ 1 \leq i \leq m$$

for some positive constant $\omega \in (0, a]$, then

$$\chi(x_1, \ldots, x_m) \leq C(m, n, a) + (m - n) \log \omega + n \log d \tag{3}$$

where $C(m, n, a)$ is a constant that depends only on $a = \max\{||x_1||_2 + 1, \ldots, ||x_m||_2 + 1\}$, m, n and $d = \max\{\deg(P_1), \ldots, \deg(P_m)\}$.

Proof. Eventually replacing each P_i by $\frac{1}{2}(P_i + P_i^*)$ we can assume from the beginning that $P_i = P_i^* \ \forall 1 \leq i \leq m$. For $R > 0$, integer $p \geq 1$ and $\epsilon > 0$ consider

$$(A_1, \ldots, A_m) \in \Gamma_R(x_1, \ldots, x_m : y_1, \ldots, y_n; p, k, \epsilon) \ .$$

If p is large enough and $\epsilon > 0$ is sufficiently small, then one can find matrices $B_1, \ldots, B_n \in \mathcal{M}_k^{sa}$ such that $||B_1||, \ldots, ||B_n|| \leq R$ and

$$||A_i - P_i(B_1, \ldots, B_n)||_2 < \omega \ \forall 1 \leq i \leq m$$

or equivalently,

$$||A_i - P_i(B_1, \ldots, B_n)||_e < \omega\sqrt{k} \ \forall 1 \leq i \leq m \ .$$

With the identifications $g = (g_1, \ldots, g_{mk^2}) : (\mathcal{M}_k^{sa})^n \cong \mathbb{R}^{nk^2} \to (\mathcal{M}_k^{sa})^m \cong \mathbb{R}^{mk^2}$, $(B_1, \ldots, B_n) = (\xi_1, \ldots, \xi_{nk^2}) \in \mathbb{R}^{nk^2}$, $g(B_1, \ldots, B_n) = (P_1(B_1, \ldots, B_n), \ldots, P_m(B_1, \ldots, B_n))$, the previous inequality becomes

$$||(A_i)_{1 \leq i \leq m} - g(\xi_1, \ldots, \xi_{nk^2})||_e < \omega\sqrt{mk}.$$

At the cost of introducing an additional variable $\xi_{nk^2+1} \in \mathbb{R}$, we can assume that the components of g are mk^2 homogeneous polynomial functions in the variables $\xi_1, \ldots, \xi_{nk^2+1}$, of degrees $\leq d$.

Let now $f_1, \ldots, f_{mk^2}$ be arbitrary homogeneous polynomial functions in $\xi_1, \ldots, \xi_{nk^2+1}$ such that $\deg(f_j) = \deg(g_j) \ \forall 1 \leq j \leq mk^2$. For every multiindex $J = (j_1, \ldots, j_{nk^2+1})$ with $1 \leq j_1 < \ldots < j_{nk^2+1} \leq mk^2$, $S_u(f_{j_1}, \ldots, f_{j_{nk^2+1}}) = \emptyset$ is equivalent to the fact that the coefficients of $f_{j_1}, \ldots, f_{j_{nk^2+1}}$ satisfy a certain system of algebraic equations. Hence the set

$$\begin{aligned} \Omega_1 = \{f = (f_1, \ldots, f_{mk^2}) \mid \deg(f_j) = \deg(g_j) \ \forall 1 \leq j \leq mk^2, \\ S_u(f_{j_1}, \ldots, f_{j_{nk^2+1}}) \neq \emptyset \ \forall J = (j_1, \ldots, j_{nk^2+1})\} \end{aligned} \tag{4}$$

is open and dense in its natural ambient linear space. Similarly, the set

$$\Omega_2 = \Big\{ f = (f_1, \dots, f_{mk^2}) \mid \deg(f_j) = \deg(g_j) \ \forall 1 \le j \le mk^2, \quad \det\left(\frac{\partial f_J}{\partial \xi}\right) \not\equiv 0 \ \forall J = (j_1, \dots, j_{nk^2+1}) \Big\} \tag{5}$$

is also open and dense in the same linear space..

The matrix df_ξ has $\binom{mk^2}{nk^2+1}$ minors of dimension $(nk^2+1)\times(nk^2+1)$ and all these minors have a nontrivial common zero only if ([vdW]) a certain system of algebraic equations in the coefficients of $f_1, \dots, f_{mk^2}$ has a solution. Moreover, not all the polynomials appearing in this system are identically equal to 0. It follows that the set

$$\Omega_3 = \big\{ f = (f_1, \dots, f_{mk^2}) \mid \deg(f_j) = \deg(g_j) \ \forall 1 \le j \le mk^2, \quad \operatorname{rank}(df_\xi) = nk^2 + 1 \ \forall \xi \in \mathbb{R}^{nk^2+1} \setminus \{0\} \big\} \tag{6}$$

contains a subset which is open and dense in the linear space previously considered. Therefore there exists an element $f \in \Omega_1 \cap \Omega_2 \cap \Omega_3$ such that $||f(\xi_1, \dots, \xi_{nk^2+1}) - g(\xi_1, \dots, \xi_{nk^2+1})||_e < \omega\sqrt{mk} \ \forall |\xi_i| \le R \ \forall 1 \le i \le nk^2 + 1$, hence $||(A_i)_{1\le i\le m} - f(\xi_1, \dots, \xi_{nk^2+1})||_e < 2\omega\sqrt{mk}$. The function f satisfies the hypothesis of Lemma 2.1 and its components are homogeneous polynomials. Moreover, it has the property that $\operatorname{dist}_e((A_i)_{1\le i\le m}, \operatorname{Range}(f)) < 2\omega\sqrt{mk}$ and it does not depend on the system $(A_i)_{1\le i\le m}$.

We have $||(A_1, \dots, A_m)||_e \le a\sqrt{mk}$ (if $\epsilon > 0$ is small enough) hence the set of matricial microstates $(A_1, \dots, A_m)$ of $(x_1, \dots, x_m)$ such that $\operatorname{dist}_e((A_1, \dots, A_m), \operatorname{Range}(f)) < 2\omega\sqrt{mk}$ is contained in the (mk^2, nk^2+1)-tube of radius $2\omega\sqrt{mk}$ around $\operatorname{Range}(f) \cap B_{mk^2}(0, (a+2\omega)\sqrt{mk})$. If B is a small ball in $\mathbb{R}^{nk^2+1} \setminus \{0\}$ and if $V_B(2\omega\sqrt{mk})$ denotes the (mk^2, nk^2+1)-tube of radius $2\omega\sqrt{mk}$ around $f(B)$, then the formula for volumes of tubes ([We]) implies

$$\operatorname{vol}_{mk^2}(V_B(2\omega\sqrt{mk})) = \operatorname{vol}_{mk^2-nk^2-1}(B_{mk^2-nk^2-1}(0,1)) \cdot \sum_{\substack{e\equiv 0 \ mod\ 2\\ 0\le e\le nk^2+1}} \frac{(2\omega\sqrt{mk})^{e+mk^2-nk^2-1} k_{B,e}}{(mk^2-nk^2+1)(mk^2-nk^2+3)\dots(mk^2-nk^2-1+e)}. \tag{7}$$

With the notations from [We] one has $k_{B,e} = \int_{f(B)} H_e ds$ and

$$H_e = \frac{1}{2^e (e/2)!} \sum_{\sigma\in\Sigma_e} \operatorname{sgn}(\sigma) \sum_{\alpha_1,\dots,\alpha_e=1}^{nk^2+1} H_{\alpha_1\alpha_2}^{\alpha_{\sigma(1)}\alpha_{\sigma(2)}} H_{\alpha_3\alpha_4}^{\alpha_{\sigma(3)}\alpha_{\sigma(4)}} \dots$$

where $H_{\alpha\beta}^{\lambda\mu}$ denotes the Riemann tensor of $f(B)$. Assuming without loss of generality that $\deg(f_j) = d \ \forall 1 \le j \le mk^2$, one can verify that each $H_{\alpha\beta}^{\lambda\mu}(f(\xi))$ is a sum of quotients of homogeneous polynomials where all numerators have degree $6(d-1)(nk^2+1) - 2d$ and all denominators have degree

$6(d-1)(nk^2+1)$, hence H_e is a rational function in ξ and in the coefficients of $f(\xi)$. Due to its intrinsic nature, H_e is independent of the embedding of Range(f) in $\mathbb{R}^{mk^2+1}$, in particular it is invariant under orthogonal transformations in $\mathbb{R}^{mk^2+1}$. Since there exist sufficiently many polynomials $f(\xi)$ such that Range(f) is flat, this entails $H_e = 0\ \forall 2 \leq e \leq nk^2+1$, $e \equiv 0 \bmod 2$. Therefore the volume of the (mk^2, nk^2+1)-tube of radius $2\omega\sqrt{mk}$ around $f(B)$ is $\mathrm{vol}_{mk^2}(V_B(2\omega\sqrt{mk})) = \mathrm{vol}_{mk^2-nk^2-1}(B_{mk^2-nk^2-1}(0,1)) \cdot (2\omega\sqrt{mk})^{mk^2-nk^2-1} \cdot \int_{f(B)} ds$ and with Lemma 2.1 and inequality

$$\frac{1}{\Gamma\left(1+\frac{nk^2+1}{2}\right)} \cdot \frac{1}{\Gamma\left(1+\frac{mk^2-nk^2-1}{2}\right)} \leq \frac{2^{\frac{mk^2}{2}}}{\Gamma\left(1+\frac{mk^2}{2}\right)} \tag{8}$$

we obtain the following estimate:

$$\begin{aligned}
&\mathrm{vol}_{mk^2}(\Gamma_R(x_1,\ldots,x_m : y_1,\ldots,y_n; p,k,\epsilon)) \leq \binom{mk^2}{nk^2+1} \cdot C(d) \\
&\cdot \mathrm{vol}_{nk^2+1}\left(B(0,(a+2\omega)\sqrt{mk})\right) \cdot \mathrm{vol}_{mk^2-nk^2-1}(B(0,1)) \\
&\cdot (2\omega\sqrt{mk})^{mk^2-nk^2-1} = \binom{mk^2}{nk^2+1} \cdot C(d) \cdot \pi^{\frac{nk^2+1}{2}} \\
&\cdot \frac{(a+2\omega)^{nk^2+1}(mk)^{\frac{nk^2+1}{2}} \pi^{\frac{mk^2-nk^2-1}{2}} (2\omega)^{mk^2-nk^2-1} (mk)^{\frac{mk^2-nk^2-1}{2}}}{\Gamma\left(1+\frac{nk^2+1}{2}\right)\Gamma\left(1+\frac{mk^2-nk^2-1}{2}\right)} \\
&\leq \binom{mk^2}{nk^2+1} \cdot C(d) \cdot \frac{\pi^{\frac{mk^2}{2}} (mk)^{\frac{mk^2}{2}} 2^{\frac{mk^2}{2}} (3a)^{nk^2+1} (2\omega)^{mk^2-nk^2-1}}{\Gamma\left(1+\frac{mk^2}{2}\right)}.
\end{aligned} \tag{9}$$

The last inequality implies further

$$\begin{aligned}
&\chi_R(x_1,\ldots,x_m : y_1,\ldots,y_n; p,k,\epsilon) \leq \frac{1}{k^2}\log\binom{mk^2}{nk^2+1} + \frac{1}{k^2}\log C(d) \\
&+\frac{m}{2}\log\pi + \left(\frac{3m}{2}-n\right)\log 2 + n\log(3a) + \frac{m}{2}\log(mk) \\
&+(m-n)\log\omega - \frac{1}{k^2}\log\Gamma\left(1+\frac{mk^2}{2}\right) + \frac{m}{2}\log k + o(1).
\end{aligned} \tag{10}$$

Note that one has $\frac{1}{k^2}\log\Gamma\left(1+\frac{mk^2}{2}\right) = \frac{m}{2}\log\frac{mk^2}{2e}+o(1)$, $C(d)\leq d^{nk^2+1}$ and $\frac{1}{k^2}\log\binom{mk^2}{nk^2+1} = m\log m - n\log n-(m-n)\log(m-n)+o(1)$, therefore

$$\begin{aligned}(11)\qquad \chi_R(x_1,\ldots,x_m:y_1,\ldots,y_n;p,k,\epsilon) &\leq m\log m - n\log n + n\log d\\ &-(m-n)\log(m-n)+\frac{m}{2}\log\pi+\left(\frac{3m}{2}-n\right)\log 2 + n\log(3a)\\ &+\frac{m}{2}\log m+\frac{m}{2}\log k+(m-n)\log\omega-\frac{m}{2}\log\frac{m}{2e}-m\log k\\ &+\frac{m}{2}\log k + o(1) = C(m,n,a)+(m-n)\log\omega+n\log d+o(1).\end{aligned}$$

By taking the appropriate limits after k,p,ϵ, we finally obtain

$$\chi_R(x_1,\ldots,x_m:y_1,\ldots,y_n)\leq C(m,n,a)+(m-n)\log\omega+n\log d,$$

and since $R>0$ is arbitrary, $\chi(x_1,\ldots,x_m:y_1,\ldots,y_n)\leq C(m,n,a)+(m-n)\log\omega+n\log d$. Recall now that $\{x_1,\ldots,x_m\}$ is a system of generators of $\mathcal{M}$, hence $\chi(x_1,\ldots,x_m)=\chi(x_1,\ldots,x_m:y_1,\ldots,y_n)$. □

Let $Y_1,\ldots,Y_n$ be noncommutative indeterminates and let

$$P(Y_1,\ldots,Y_n)=\sum_{k=0}^{\infty}\sum_{1\leq i_1,\ldots,i_k\leq n} a_{i_1\ldots i_k}Y_{i_1}\ldots Y_{i_k}$$

be a noncommutative power series in $Y_1,\ldots,Y_n$, with complex coefficients. Following [Vo2], we say that $R>0$ is a radius of convergence of P if

$$\sum_{k=0}^{\infty}\sum_{1\leq i_1,\ldots,i_k\leq n}|a_{i_1\ldots i_k}|R^k<\infty\ .$$

It is well-known from the theory of power series that if $0<R_0<R$, then

$$\sum_{k=q+1}^{\infty}\sum_{1\leq i_1,\ldots,i_k\leq n}|a_{i_1\ldots i_k}|R_0^k = O\left(\left(\frac{R_0}{R}\right)^{q+1}\right)\ .$$

Theorem 2.3 is basically Corollary 6.12 in [Vo2], with the observation that the freeness of $\{x_1,\ldots,x_m\}$ assumed there has been dropped.

Theorem 2.3. *Let $x_1,\ldots,x_m$ and $y_1,\ldots,y_n$ be self-adjoint noncommutative random variables in a II_1-factor $(\mathcal{M},\tau)$ such that $y_1,\ldots,y_n\in\{x_1,\ldots,x_m\}''$ and $\chi(x_1,\ldots,x_m)>-\infty$. If $x_i=P_i(y_1,\ldots,y_n)$ $\forall 1\leq i\leq m$, where $(P_i)_{1\leq i\leq m}$ are noncommutative power series having a common radius of convergence $R>b=\max\{||y_1||,\ldots,||y_n||\}$, then $n\geq m$.*

Proof. Suppose that $m>n$. For $1\leq i\leq m$, x_i is a noncommutative power series of $y_1,\ldots,y_n$ i.e.,

$$x_i=\sum_{k=0}^{\infty}\sum_{1\leq i_1,\ldots,i_k\leq n}a^{(i)}_{i_1\ldots i_k}y_{i_1}\ldots y_{i_k}\ .$$

For every integer $q \geq 0$, $P_{i,q}(y_1, \ldots, y_n) := \sum_{k=0}^{q} \sum_{1 \leq i_1, \ldots, i_k \leq n} a^{(i)}_{i_1 \ldots i_k} y_{i_1} \ldots y_{i_k}$ is a noncommutative polynomial of degree $\leq q$ and moreover

$$
\begin{aligned}
(12) \qquad ||x_i - P_{i,q}(y_1, \ldots, y_n)||_2 &= \left|\left| \sum_{k=q+1}^{\infty} \sum_{1 \leq i_1, \ldots, i_k \leq n} a^{(i)}_{i_1 \ldots i_k} y_{i_1} \ldots y_{i_k} \right|\right|_2 \\
&\leq \sum_{k=q+1}^{\infty} \sum_{1 \leq i_1, \ldots, i_k \leq n} |a^{(i)}_{i_1 \ldots i_k}| b^k = O\left(\left(\frac{b}{R} \right)^{q+1} \right).
\end{aligned}
$$

The estimate of free entropy from Proposition 2.2 implies $\chi(x_1, \ldots, x_m) \leq C(m, n, a) + (m - n) \log \left(\frac{b}{R}\right)^{q+1} + n \log q + O(1)$ and letting q tend to ∞, one obtains that $\chi(x_1, \ldots, x_m) = -\infty$, contradiction. $\square$

Let $\mathcal{N}$ be a $*$-algebra in a W^*-probability space $(\mathcal{M}, \tau)$. Suppose that $\mathcal{N}$ is finitely generated and let $\{x_1, \ldots, x_m\}$ be a system of self-adjoint generators. Let also $\{y_1, \ldots, y_n\}$ be another set of self-adjoint elements that generate $\mathcal{N}$ *algebraically* as a $*$-algebra. In particular, there exist noncommutative polynomials $(P_i)_{1 \leq i \leq m}$ such that $x_i = P_i(y_1, \ldots, y_n)$ $\forall 1 \leq i \leq m$. In this context, Corollary 2.4 is an immediate consequence of Theorem 2.3.

Corollary 2.4. *If $\chi(x_1, \ldots, x_m) > -\infty$ and $*$-alg$\{y_1, \ldots, y_n\} = *$-alg$\{x_1, \ldots, x_m\}$ then $n \geq m$, so any 2 systems of self-adjoint elements with finite free entropy that generate $\mathcal{N}$ algebraically as a $*$-algebra have the same cardinality.*

D. Voiculescu proved in [Vo5] that the modified free entropy dimension ([Vo3]) of a finite set of self-adjoint elements that generate algebraically a $*$-algebra $\mathcal{N}$ is independent of the set of generators. Recall ([Vo3]) the definition of the modified free entropy dimension:

$$
\delta_0(x_1, \ldots, x_m) = m + \limsup_{\omega \to 0} \frac{\chi(x_1 + \omega s_1, \ldots, x_1 + \omega s_m : s_1, \ldots, s_m)}{|\log \omega|},
$$

where $\{s_1, \ldots, s_m\}$ is a semicircular system free from $\{x_1, \ldots, x_m\}$. One has $\delta_0(x_1, \ldots, x_m) \leq m$ in general, and also $0 \leq \delta_0(x_1, \ldots, x_m)$ if $\{x_1, \ldots, x_m\} \subset \mathcal{L}(\mathbb{F}_p)$ for some p. Considering two sets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ of self-adjoint elements that generate algebraically the $*$-algebra $\mathcal{N}$ and noticing that $\{y_1, \ldots, y_n\} \subset \{x_1 + \omega s_1, \ldots, x_1 + \omega s_m, s_1, \ldots, s_m\}''$, one has

$$
\begin{aligned}
(13) \qquad \delta_0(x_1, \ldots, x_m) &= m \\
&+ \limsup_{\omega \to 0} \frac{\chi(x_1 + \omega s_1, \ldots, x_1 + \omega s_m : s_1, \ldots, s_m, y_1, \ldots, y_n)}{|\log \omega|} \\
&\leq m + \limsup_{\omega \to 0} \frac{\chi(x_1 + \omega s_1, \ldots, x_1 + \omega s_m : y_1, \ldots, y_n)}{|\log \omega|}.
\end{aligned}
$$

Also, $||x_i + \omega s_i - P_i(y_1, \ldots, y_n)|| = ||\omega s_i|| \leq \omega \; \forall 1 \leq i \leq m$, and with Proposition 2.2 we obtain

$$\begin{aligned}(14)\quad \delta_0(x_1, \ldots, x_m) &\leq m \\ &+ \limsup_{\omega\to 0} \frac{C(m,n,a) + (m-n)\log\omega + n\log d}{|\log\omega|} \leq m + n - m = n,\end{aligned}$$

where $a = \max\{||x_1||_2 + 1, \ldots, ||x_m||_2 + 1, ||y_1||_2 + 1, \ldots, ||y_n||_2 + 1\}$ and $d = \max\{\deg(P_i) \,|\, 1 \leq i \leq m\}$. In particular, if there exists a set $\{y_1, \ldots, y_n\}$ with $\delta_0(y_1, \ldots, y_n) = n$ which generates $\mathcal{N}$ algebraically, then $\sup\{\delta_0(x_1, \ldots, x_m) \mid *\text{-alg}\{x_1, \ldots, x_m\} = \mathcal{N}\} = n$.

3. Indecomposability over nonprime subfactors

In this section we prove that the free group factor $\mathcal{L}(\mathbb{F}_n)$ does not admit an asymptotic decomposition of the form

$$\lim_{\omega\to 0}{}^{||\cdot||_2} \sum_{\substack{1\leq j_1,\ldots,j_{t+1}\leq f \\ 1\leq t\leq d}} \mathcal{N}^\omega_{j_1}\mathcal{Z}^\omega\mathcal{N}^\omega_{j_2}\mathcal{Z}^\omega \ldots \mathcal{N}^\omega_{j_t}\mathcal{Z}^\omega\mathcal{N}^\omega_{j_{t+1}},$$

where $\{\mathcal{Z}^\omega \subset \mathcal{L}(\mathbb{F}_n)\}_\omega$ are subsets with p self-adjoint elements, $\{\mathcal{N}^\omega_1, \ldots, \mathcal{N}^\omega_f\}$ are nonprime subfactors of $\mathcal{L}(\mathbb{F}_n)$, $d \geq 1$ is an arbitrary integer, and $n \geq p + 2f + 1$. A nonprime II_1-factor is just a factor isomorphic to the tensor product of two factors of type II_1. For free group subfactors one has the following: if $n \geq p + 2f + 2$ and $\mathcal{P} \subset \mathcal{L}(\mathbb{F}_n)$ is a subfactor of finite index, then $\mathcal{P}$ does not admit such an asymptotic decomposition either. In particular, the hyperfinite dimension of $\mathcal{L}(\mathbb{F}_n)$ is $\geq [\frac{n-2}{2}] + 1$ and the hyperfinite dimension of $\mathcal{P}$ is $\geq [\frac{n-3}{2}] + 1$. For $n = \infty$ this settles a conjecture of L. Ge and S. Popa ([GePo]): the hyperfinite dimension of free group factors is infinite. The definitions of hyperfinite dimension and of asymptotic decomposition over nonprime subfactors are given next.

Definition 3.1. ([GePo]) If $\mathcal{M}$ is a type II_1-factor, then the hyperfinite dimension of $\mathcal{M}$, denoted $\ell_h(\mathcal{M})$, is by definition the smallest positive integer $f \in \mathbb{N}$ with the property that there exist hyperfinite subalgebras $\mathcal{R}_1, \ldots, \mathcal{R}_f \subset \mathcal{M}$ such that $\overline{\mathrm{sp}}^w \mathcal{R}_1\mathcal{R}_2 \ldots \mathcal{R}_f = \mathcal{M}$. If there is no such positive integer f, then by definition, $\ell_h(\mathcal{M}) = +\infty$.

Definition 3.2. A type II_1-factor $\mathcal{M}$ admits an asymptotic decomposition over nonprime subfactors, denoted

$$\lim_{\omega\to 0}{}^{||\cdot||_2} \sum_{\substack{1\leq j_1,\ldots,j_{t+1}\leq f \\ 1\leq t\leq d}} \mathcal{N}^\omega_{j_1}\mathcal{Z}^\omega\mathcal{N}^\omega_{j_2}\mathcal{Z}^\omega \ldots \mathcal{N}^\omega_{j_t}\mathcal{Z}^\omega\mathcal{N}^\omega_{j_{t+1}}\,,$$

provided that $\forall n \geq 1 \; \forall x_1, \ldots, x_n \in \mathcal{M} \; \forall \omega > 0 \; \exists \mathcal{N}^\omega_1 = \mathcal{N}_1(x_1, \ldots, x_n; \omega), \ldots, \mathcal{N}^\omega_f = \mathcal{N}_f(x_1, \ldots, x_n; \omega)$ nonprime subfactors of $\mathcal{M}$ $\exists \mathcal{Z}^\omega = \mathcal{Z}(x_1,$

$\ldots, x_n; \omega) \subset \mathcal{M}$ containing p self-adjoint elements, such that

$$\mathrm{dist}_{||\cdot||_2}\left(x_j, \sum_{\substack{1\le j_1,\ldots,j_{t+1}\le f\\ 1\le t\le d}} \mathcal{N}^{\omega}_{j_1}\mathcal{Z}^{\omega}\mathcal{N}^{\omega}_{j_2}\mathcal{Z}^{\omega}\ldots\mathcal{N}^{\omega}_{j_t}\mathcal{Z}^{\omega}\mathcal{N}^{\omega}_{j_{t+1}}\right) < \omega \ \forall 1\le j\le n.$$

If $\mathcal{L}(\mathbb{F}_n)$ admitted an asymptotic decomposition over nonprime subfactors (Definition 3.2), then the situation described in Proposition 3.4 (with $\mathcal{M} = \mathcal{L}(\mathbb{F}_n)$) would take place for an arbitrary $\omega > 0$, since any II_1-factor is generated by its projections of given trace ($\frac{1}{2}$, for example):

Lemma 3.3. *([KR]) Any type II_1-factor $\mathcal{M}$ with separable predual is generated by a countable family of projections of given trace.*

Proof. Every II_1-factor with separable predual is generated by a countable family of abelian subalgebras, so there exist $\mathcal{A}_1, \mathcal{A}_2, \ldots$ abelian subalgebras of $\mathcal{M}$ that generate $\mathcal{M}$ as a von Neumann algebra. If necessary, one can replace each $\mathcal{A}_n$ by a maximal abelian subalgebra of $\mathcal{M}$ which contains it, hence we can assume that $\mathcal{A}_n$ is a maximal abelian subalgebra of $\mathcal{M}$ $\forall 1 \le n < \infty$. Being a maximal abelian subalgebra of a type II_1-factor, $\mathcal{A}_n$ has no atoms and thus it is generated by a countable subset of projections of given trace, $\forall 1 \le n < \infty$. □

Proposition 3.4. *Let $z_1, \ldots, z_p$ be self-adjoint elements of a II_1-factor $\mathcal{M}$ and let $(\mathcal{N}_v)_{1\le v\le f}$ be a family of subfactors of $\mathcal{M}$. Assume that $\mathcal{N}_v = \mathcal{R}_1^{(v)} \vee \mathcal{R}_2^{(v)} \simeq \mathcal{R}_1^{(v)} \otimes \mathcal{R}_2^{(v)}$ where $\mathcal{R}_1^{(v)}, \mathcal{R}_2^{(v)}$ are II_1-factors and assume that $x_1, \ldots, x_n$ are self-adjoint generators of $\mathcal{M}$. Assume moreover that there exist projections of trace $\frac{1}{2}$, $p_1^{(v)}, \ldots, p_{r_v}^{(v)} \in \mathcal{R}_2^{(v)}$, $q_1^{(v)}, \ldots, q_{s_v}^{(v)} \in \mathcal{R}_1^{(v)}$ and complex noncommutative polynomials $(\phi_j)_{1\le j\le n}$ of degrees $\le d$ (where $d \ge 1$ is fixed) in the variables $(z_u)_{1\le u\le p}$ such that*

$$\left|\left| x_j - \phi_j\left((p_i^{(v)})_{\substack{1\le i\le r_v\\ 1\le v\le f}}, (q_l^{(v)})_{\substack{1\le l\le s_v\\ 1\le v\le f}}, (z_u)_{1\le u\le p}\right)\right|\right|_2 < \omega,\ j = 1, \ldots, n, \tag{1}$$

where $\omega \in (0, a]$ is a given positive number, and such that in all the monomials of each ϕ_j the projections $p_i^{(v)}, q_l^{(v)}$ and $p_k^{(w)}, q_s^{(w)}$ are separated by some z_u if $v \ne w$. Then

$$\chi(x_1, \ldots, x_n) \le C(n, p, a, d, f) + (n - p - 2f)\log\omega, \tag{2}$$

where $a = \max\{||x_j||_2 + 1 | 1 \le j \le n\}$ and $C(n, p, a, d, f)$ is a constant that depends only on n, p, a, d, f.

Proof. All variables involved are self-adjoint so we can assume that $\phi_j = \phi_j^*$ $\forall 1 \le j \le n$. Fix an integer $k_0 \ge 1$ and let $R > 0$. Let $\mathcal{M}_{k_0}(\mathbb{C}) \cong \mathcal{M}_1^{(v)} \subset \mathcal{R}_1^{(v)}$, $\mathcal{M}_{k_0}(\mathbb{C}) \cong \mathcal{M}_2^{(v)} \subset \mathcal{R}_2^{(v)}$ and $\{e_{jl}^{(v)}\}_{j,l}$, $\{f_{jl}^{(v)}\}_{j,l}$ be matrix units for

$\mathcal{M}_1^{(v)}$ and $\mathcal{M}_2^{(v)}$ respectively. If

$$\left((A_j)_{1\le j\le n}, (G_i^{(v)})_{\substack{1\le i\le r_v\\ 1\le v\le f}}, (H_l^{(v)})_{\substack{1\le l\le s_v\\ 1\le v\le f}}, \{E_{jl}^{(v)}\}_{j,l,v}, \{F_{jl}^{(v)}\}_{j,l,v}, (Z_u)_{1\le u\le p}\right)$$

is an arbitrary microstate in the set of matricial microstates

$$\Gamma_R\Bigg((x_j)_{1\le j\le n}, (p_i^{(v)})_{\substack{1\le i\le r_v\\ 1\le v\le f}}, (q_l^{(v)})_{\substack{1\le l\le s_v\\ 1\le v\le f}}, \{e_{jl}^{(v)}\}_{j,l,v}, \{f_{jl}^{(v)}\}_{j,l,v}, (z_u)_{1\le u\le p}; m, k, \epsilon\Bigg) \tag{3}$$

and if m is large and ϵ is small enough, then

$$\left|\left|A_j - \phi_j\left((G_i^{(v)})_{\substack{1\le i\le r_v\\ 1\le v\le f}}, (H_l^{(v)})_{\substack{1\le l\le s_v\\ 1\le v\le f}}, (Z_u)_{1\le u\le p}\right)\right|\right|_2 < \omega,\ j = 1, \ldots, n.$$

Let $\delta > 0$ and write $k = k_0^2 t + w$ for some integers w, t with $0 \le w \le k_0^2 - 1$. If m, ϵ are suitably chosen, then there exist $\mathcal{M}_1^{(v)} \cong \tilde{\mathcal{M}}_1^{(v)} \subset \mathcal{M}_k(\mathbb{C})$, $\mathcal{M}_2^{(v)} \cong \tilde{\mathcal{M}}_2^{(v)} \subset \mathcal{M}_k(\mathbb{C})$ (not necessarily unital inclusions) and matrix units $\{\tilde{E}_{jl}^{(v)}\}_{j,l,v} \subset \tilde{\mathcal{M}}_1^{(v)}$, $\{\tilde{F}_{jl}^{(v)}\}_{j,l,v} \subset \tilde{\mathcal{M}}_2^{(v)}$, such that

$$\left|\left|\tilde{E}_{jl}^{(v)} - E_{jl}^{(v)}\right|\right|_2 < \delta,\ \left|\left|\tilde{F}_{jl}^{(v)} - F_{jl}^{(v)}\right|\right|_2 < \delta\ \forall 1 \le j, l \le k_0,$$

and $\tilde{\mathcal{M}}_1^{(v)} \subset \left(\tilde{\mathcal{M}}_2^{(v)}\right)' \cap \mathcal{M}_k(\mathbb{C})$. The relative commutants of $\tilde{\mathcal{M}}_1^{(v)}$ and $\tilde{\mathcal{M}}_2^{(v)}$ in $\mathcal{M}_k(\mathbb{C})$ satisfy $\left(\tilde{\mathcal{M}}_1^{(v)}\right)' \cap \mathcal{M}_k(\mathbb{C}) \cong (\mathcal{M}_{k_0}(\mathbb{C}) \otimes 1 \otimes \mathcal{M}_t(\mathbb{C})) \oplus \mathcal{M}_w(\mathbb{C})$ and $\left(\tilde{\mathcal{M}}_2^{(v)}\right)' \cap \mathcal{M}_k(\mathbb{C}) \cong (1 \otimes \mathcal{M}_{k_0}(\mathbb{C}) \otimes \mathcal{M}_t(\mathbb{C})) \oplus \mathcal{M}_w(\mathbb{C})$. Let $\eta^{(v)}(x, \{e_{jl}^{(v)}\}_{j,l}) := \frac{1}{k_0}\sum_{j,l=1}^{k_0} e_{jl}^{(v)} x e_{lj}^{(v)} \in \mathbb{C} < X_1, \ldots, X_{k_0^2+1} >$ be the polynomial in $k_0^2 + 1$ indeterminates that gives the conditional expectation $E_{(\mathcal{M}_1^{(v)})' \cap \mathcal{M}} : \mathcal{M} \to (\mathcal{M}_1^{(v)})' \cap \mathcal{M}$, $E_{(\mathcal{M}_1^{(v)})' \cap \mathcal{M}}(x) = \eta^{(v)}(x, \{e_{jl}^{(v)}\}_{j,l})$. Then $G_1^{(v,1)} := \eta^{(v)}(G_1^{(v)}, \{\tilde{E}_{jl}^{(v)}\}_{j,l}) \in \left(\tilde{\mathcal{M}}_1^{(v)}\right)' \cap \mathcal{M}_k(\mathbb{C})$ and since $p_1^{(v)} = E_{(\mathcal{M}_1^{(v)})' \cap \mathcal{M}}(p_1^{(v)}) = \eta^{(v)}(p_1^{(v)}, \{e_{jl}^{(v)}\}_{j,l})$ it follows that

$$\left|\tau_k\left((G_1^{(v,1)})^l\right) - \tau\left((p_1^{(v)})^l\right)\right| < \delta_1,\ \forall 1 \le l \le m_1$$

for any given δ_1, m_1, provided that ϵ, δ are small and m is large enough. For suitable m_1, δ_1 there exists a projection $P_1^{(v,1)} \in \left(\tilde{\mathcal{M}}_1^{(v)}\right)' \cap \mathcal{M}_k(\mathbb{C})$ of rank $\left[\frac{k_0 t + w}{2}\right]$ such that $||P_1^{(v,1)} - G_1^{(v,1)}||_2 < \delta_2$. Then $||G_1^{(v)} - P_1^{(v,1)}||_2 \le ||G_1^{(v)} - G_1^{(v,1)}||_2 + ||G_1^{(v,1)} - P_1^{(v,1)}||_2 < 2\delta_2$ since $||G_1^{(v)} - G_1^{(v,1)}||_2 < \delta_2$ for convenient m, ϵ, δ. With this procedure we can find projections $P_1^{(v,1)}, \ldots, P_{r_v}^{(v,1)} \in \left(\tilde{\mathcal{M}}_1^{(v)}\right)' \cap \mathcal{M}_k(\mathbb{C})$ and $Q_1^{(v,1)}, \ldots, Q_{s_v}^{(v,1)} \in \left(\tilde{\mathcal{M}}_2^{(v)}\right)' \cap \mathcal{M}_k(\mathbb{C})$, all of rank

$\left[\frac{k_0t+w}{2}\right]$, such that $||G_i^{(v)} - P_i^{(v,1)}||_2 < 2\delta_2$ and $||H_j^{(v)} - Q_j^{(v,1)}||_2 < 2\delta_2$ for all indices i, j, v. Moreover,

$$\left|\left|A_j - \phi_j\left((P_i^{(v,1)})_{\substack{1\le i\le r_v\\1\le v\le f}}, (Q_l^{(v,1)})_{\substack{1\le l\le s_v\\1\le v\le f}}, (Z_u)_{1\le u\le p}\right)\right|\right|_2 < \omega\,\forall\, 1\le j\le n$$

if we choose a sufficiently small $\delta_2 > 0$. Let $\mathcal{G}_1^{(v)}(k) \subset \left(\tilde{\mathcal{M}}_1^{(v)}\right)' \cap \mathcal{M}_k(\mathbb{C})$ and $\mathcal{G}_2^{(v)}(k) \subset \left(\tilde{\mathcal{M}}_2^{(v)}\right)' \cap \mathcal{M}_k(\mathbb{C})$ be 2 fixed copies of the Grassmann manifold $\mathcal{G}\left(k_0t+w, \left[\frac{k_0t+w}{2}\right]\right)$ (projections in $\mathcal{M}_{k_0t+w}(\mathbb{C})$, of rank $\left[\frac{k_0t+w}{2}\right]$). There exists a unitary $U^{(v)} \in \mathcal{U}(k)$ such that $U^{(v)}P_1^{(v,1)}U^{(v)*}, \ldots, U^{(v)}P_{r_v}^{(v,1)}U^{(v)*} \in \mathcal{G}_1^{(v)}(k)$ and $U^{(v)}Q_1^{(v,1)}U^{(v)*}, \ldots, U^{(v)}Q_{s_v}^{(v,1)}U^{(v)*} \in \mathcal{G}_2^{(v)}(k)$. The previous inequality becomes

$$\begin{aligned}(4)\quad &\left|\left|A_j - \phi_j\Big((U^{(v)}P_i^{(v,1)}U^{(v)*})_{\substack{1\le i\le r_v\\1\le v\le f}}, (U^{(v)}Q_l^{(v,1)}U^{(v)*})_{\substack{1\le l\le s_v\\1\le v\le f}}, (Z_u)_{1\le u\le p},\right.\right.\\ &\qquad\left.\left.(\mathrm{Re}(U^{(v)}), \mathrm{Im}(U^{(v)}))_{1\le v\le f}\Big)\right|\right|_2 < \omega\,\forall\, 1\le j\le n.\end{aligned}$$

The euclidean norm on $\mathcal{M}_k^{sa}$ induces a $\mathcal{U}(k_0t+w)$-invariant metric on the manifold $\mathcal{G}\left(k_0t+w, \left[\frac{k_0t+w}{2}\right]\right)$ and if $\{P_a\}_{a\in A(k)}$ is a minimal θ-net in the manifold with respect to this metric, then ([Sz]) $|A(k)| \le \left(\frac{Ch_k}{\theta}\right)^{g_k}$ where C is a universal constant, $g_k = 2\left[\frac{k_0t+w}{2}\right]\cdot\left(k_0t+w-\left[\frac{k_0t+w}{2}\right]\right)$ is the dimension of $\mathcal{G}\left(k_0t+w, \left[\frac{k_0t+w}{2}\right]\right)$ and $h_k \le \sqrt{2k}$ is the diameter of the Grassmann manifold $\mathcal{G}\left(k_0t+w, \left[\frac{k_0t+w}{2}\right]\right)$ in $\mathcal{M}_k^{sa}$. There exist $\alpha := (a_1^{(v)}, \ldots, a_{r_v}^{(v)})_{1\le v\le f}$ and $\beta := (b_1^{(v)}, \ldots, b_{s_v}^{(v)})_{1\le v\le f}$ with entries from $A(k)$ such that

$$\left|\left|P_{a_i^{(v)}}^{(v)} - U^{(v)}P_i^{(v,1)}U^{(v)*}\right|\right|_e \le \theta \text{ and } \left|\left|P_{b_l^{(v)}}^{(v)} - U^{(v)}Q_l^{(v,1)}U^{(v)*}\right|\right|_e \le \theta$$

for all $1\le i\le r_v$, $1\le l\le s_v$, $1\le v\le f$. The polynomials $(\phi_j)_{1\le j\le n}$ are in particular Lipschitz functions hence there exists a constant $D = D\left((\phi_j)_{1\le j\le n}, R\right) > 0$ (note that $|\alpha| = r_1+\ldots+r_f$ and $|\beta| = s_1+\ldots+s_f$) such that

$$\begin{aligned}(5)\qquad &\left|\left|\phi_j(V_1,\ldots,V_{|\alpha|+|\beta|+p+2f}) - \phi_j(W_1,\ldots,W_{|\alpha|+|\beta|+p+2f})\right|\right|_e\\ &\quad\le D\left|\left|(V_1,\ldots,V_{|\alpha|+|\beta|+p+2f}) - (W_1,\ldots,W_{|\alpha|+|\beta|+p+2f})\right|\right|_e\end{aligned}$$

for all $1 \leq j \leq n$ and all $V_1, \ldots, V_{|\alpha|+|\beta|+p+2f}, W_1, \ldots, W_{|\alpha|+|\beta|+p+2f} \in \{V \in M_k \mid ||V|| \leq R\}$. We have then

$$(6)\quad \left|\left|A_j - \phi_j\Big((P_a)_{a\in\alpha}, (P_b)_{b\in\beta}, (Z_u)_{1\leq u\leq p}, \big(\mathrm{Re}(U^{(v)}), \mathrm{Im}(U^{(v)})\big)_{1\leq v\leq f}\Big)\right|\right|_e$$
$$< \omega\sqrt{k} + D\left|\left|\Big((U^{(v)}P_i^{(v,1)}U^{(v)*})_{\substack{1\leq i\leq r_v\\ 1\leq v\leq f}}, (U^{(v)}Q_l^{(v,1)}U^{(v)*})_{\substack{1\leq l\leq s_v\\ 1\leq v\leq f}},\right.\right.$$
$$(Z_u)_{1\leq u\leq p}, \big(\mathrm{Re}(U^{(v)}), \mathrm{Im}(U^{(v)})\big)_{1\leq v\leq f}\Big) - \Big((P_a)_{a\in\alpha}, (P_b)_{b\in\beta},$$
$$\left.\left.(Z_u)_{1\leq u\leq p}, \big(\mathrm{Re}(U^{(v)}), \mathrm{Im}(U^{(v)})\big)_{1\leq v\leq f}\Big)\right|\right|_e < \omega\sqrt{k}$$
$$+ D\theta\sqrt{|\alpha|+|\beta|} = 2\omega\sqrt{k},$$

if we choose $\theta := \frac{\omega}{D}\sqrt{\frac{k}{|\alpha|+|\beta|}}$. Define $F_{\alpha,\beta} : (\mathcal{M}_k^{sa})^{p+2f} \to (\mathcal{M}_k^{sa})^n$ by

$$(7)\quad F_{\alpha,\beta}\left((W_u)_{1\leq u\leq p}, (W_1^{(v)}, W_2^{(v)})_{1\leq v\leq f}\right)$$
$$= \left(\phi_j((P_a)_{a\in\alpha}, (P_b)_{b\in\beta}, (W_u)_{1\leq u\leq p}, (W_1^{(v)}, W_2^{(v)})_{1\leq v\leq f})\right)_{1\leq j\leq n},$$

and note that $\mathrm{dist}_e((A_j)_{1\leq j\leq n}, \mathrm{Range}(F_{\alpha,\beta})) < 2\omega\sqrt{nk}$. Note also that all the components of $F_{\alpha,\beta}$ are polynomial functions of degrees $\leq 3d+2$. Use now Lemma 2.1 as in the proof of Proposition 2.2 to obtain the estimates:

$$(8)\quad \mathrm{vol}_{nk^2}\bigg(\Gamma_R\Big((x_j)_{1\leq j\leq n} : (p_i^{(v)})_{\substack{1\leq i\leq r_v\\ 1\leq v\leq f}}, (q_l^{(v)})_{\substack{1\leq l\leq s_v\\ 1\leq v\leq f}}, \{e_{jl}^{(v)}\}_{j,l,v}, \{f_{jl}^{(v)}\}_{j,l,v},$$
$$(z_u)_{1\leq u\leq p}; m, k, \epsilon\Big)\bigg) \leq \left(\left(\frac{Ch_k}{\theta}\right)^{g_k}\right)^{|\alpha|+|\beta|} \cdot \binom{nk^2}{(p+2f)k^2} \cdot C(d)$$
$$\cdot \mathrm{vol}_{(p+2f)k^2}\left(B(0, (a+2\omega)\sqrt{nk})\right) \cdot \mathrm{vol}_{nk^2-(p+2f)k^2}\left(B(0, 2\omega\sqrt{nk})\right)$$
$$= \left(\frac{CDh_k}{\omega}\sqrt{\frac{|\alpha|+|\beta|}{k}}\right)^{(|\alpha|+|\beta|)g_k} \cdot \binom{nk^2}{(p+2f)k^2} \cdot C(d)$$
$$\cdot \frac{(\pi nk)^{\frac{(p+2f)k^2}{2}}(2\omega+a)^{(p+2f)k^2}}{\Gamma\left(1+\frac{(p+2f)k^2}{2}\right)} \cdot \frac{(\pi nk)^{\frac{nk^2-(p+2f)k^2}{2}}(2\omega)^{nk^2-(p+2f)k^2}}{\Gamma\left(1+\frac{nk^2-(p+2f)k^2}{2}\right)}.$$

The above estimate, the inequality (8) on pag. 8, and the inequalities

$$(9)\quad h_k \leq \sqrt{2k}, 0 < \omega \leq a, g_k = 2\left[\frac{k_0t+w}{2}\right]\left(k_0t + w - \left[\frac{k_0t+w}{2}\right]\right) \leq 2$$
$$\cdot \frac{k_0t+w}{2} \cdot \left(k_0t + w - \frac{k_0t+w}{2}\right) = \frac{(k_0t+w)^2}{2} = \frac{(k+k_0w-w)^2}{2k_0^2},$$

together with $C(d) \leq (3d+2)^{(p+2f)k^2}$ imply

$$\text{(10)}\quad \mathrm{vol}_{nk^2}\Bigg(\Gamma_R\Bigg((x_j)_{1\leq j\leq n} : (p_i^{(v)})_{\substack{1\leq i\leq r_v\\ 1\leq v\leq f}}, (q_l^{(v)})_{\substack{1\leq l\leq s_v\\ 1\leq v\leq f}}, \{e_{jl}^{(v)}\}_{j,l,v}, \{f_{jl}^{(v)}\}_{j,l,v},$$
$$(z_u)_{1\leq u\leq p}; m,k,\epsilon\Bigg)\Bigg) \leq \left(\frac{CD\sqrt{2(|\alpha|+|\beta|)}}{\omega}\right)^{\frac{(k+k_0w-w)^2}{2k_0^2}(|\alpha|+|\beta|)}$$
$$\cdot\frac{2^{\frac{nk^2}{2}}(\pi nk)^{\frac{nk^2}{2}}(3a)^{(p+2f)k^2}(2\omega)^{(n-p-2f)k^2}}{\Gamma\left(1+\frac{nk^2}{2}\right)}$$
$$\cdot\binom{nk^2}{(p+2f)k^2}\cdot(3d+2)^{(p+2f)k^2}$$

therefore

$$\text{(11)}\quad \frac{1}{k^2}\chi_R\Bigg((x_j)_{1\leq j\leq n} : (p_i^{(v)})_{\substack{1\leq i\leq r_v\\ 1\leq v\leq f}}, (q_l^{(v)})_{\substack{1\leq l\leq s_v\\ 1\leq v\leq f}}, \{e_{jl}^{(v)}\}_{j,l,v}, \{f_{jl}^{(v)}\}_{j,l,v},$$
$$(z_u)_{1\leq u\leq p}; m,k,\epsilon\Bigg) + \frac{n}{2}\log k \leq C(n,p,a,d,f) + n\log k$$
$$+\frac{|\alpha|+|\beta|}{2k_0^2}\left(1+\frac{k_0w-w}{k}\right)^2\log\frac{CD\sqrt{2(|\alpha|+|\beta|)}}{\omega}$$
$$+(n-p-2f)\log\omega - \frac{1}{k^2}\log\Gamma\left(1+\frac{nk^2}{2}\right) + \frac{1}{k^2}\log\binom{nk^2}{(p+2f)k^2}.$$

Use $\frac{1}{k^2}\log\binom{nk^2}{(p+2f)k^2} = n\log n - (p+2f)\log(p+2f) - (n-p-2f)\log(n-p-2f) + o(1)$ and Stirling's formula $\frac{1}{k^2}\log\Gamma\left(1+\frac{nk^2}{2}\right) = \frac{n}{2}\log\frac{nk^2}{2e} + o(1)$ to conclude

$$\text{(12)}\quad \chi_R\Bigg((x_j)_{1\leq j\leq n} : (p_i^{(v)})_{\substack{1\leq i\leq r_v\\ 1\leq v\leq f}}, (q_l^{(v)})_{\substack{1\leq l\leq s_v\\ 1\leq v\leq f}}, \{e_{jl}^{(v)}\}_{j,l,v}, \{f_{jl}^{(v)}\}_{j,l,v},$$
$$(z_u)_{1\leq u\leq p}; m,\epsilon\Bigg) \leq \frac{|\alpha|+|\beta|}{2k_0^2}\log(CD\sqrt{2(|\alpha|+|\beta|)})$$
$$+C(n,p,a,d,f) + \left(n-p-2f-\frac{|\alpha|+|\beta|}{2k_0^2}\right)\log\omega.$$

The last inequality shows that the free entropy of $\{x_1,\dots,x_n\}$ does not exceed $C(n,p,a,d,f)+(n-p-2f)\log\omega$ since k_0 is an arbitrary integer, R is an arbitrary positive number and $x_1,\dots,x_n$ generate M. $\square$

3.1. **Hyperfinite dimension of free group factors.**

Theorem 3.5. *If $n \geq p+2f+1$, then the free group factor $\mathcal{L}(\mathbb{F}_n)$ can not be asymptotically decomposed as*

$$\lim_{\omega\to 0}{}^{\|\cdot\|_2}\sum_{\substack{1\leq j_1,\dots,j_{t+1}\leq f\\ 1\leq t\leq d}}\mathcal{N}_{j_1}^{\omega}\mathcal{Z}^{\omega}\mathcal{N}_{j_2}^{\omega}\mathcal{Z}^{\omega}\dots\mathcal{N}_{j_t}^{\omega}\mathcal{Z}^{\omega}\mathcal{N}_{j_{t+1}}^{\omega}$$

where $\{\mathcal{Z}^\omega \subset \mathcal{L}(\mathbb{F}_n)\}_\omega$ *contain* p *self-adjoint elements,* $\{\mathcal{N}_1^\omega, \ldots, \mathcal{N}_f^\omega\}_\omega$ *are nonprime subfactors of* $\mathcal{L}(\mathbb{F}_n)$*, and* $d \geq 1$ *is an integer.*

Proof. Suppose first that $\infty > n \geq p + 2f + 1$ and consider a semicircular system $\{x_1, \ldots, x_n\}$ that generates $\mathcal{L}(\mathbb{F}_n)$ as a von Neumann algebra. If the assertion were true then one could find for every $\omega > 0$ noncommutative polynomials and projections as in Proposition 3.4, satisfying the inequalities (1). But then the estimate of the free entropy (2) would imply that $\chi(x_1, \ldots, x_n) = -\infty$ if one makes ω tend to 0, contradiction.

If $n = \infty$ then $\mathcal{L}(\mathbb{F}_\infty)$ is generated by an infinite semicircular system $\{x_t\}_{t\geq 1}$. If we fix an integer $k \geq p + 2f + 1$, then we can approximate $x_1, \ldots, x_k$ by polynomials $(\phi_j)_{1\leq j\leq k}$ as in (1) and so one has the estimate of the modified free entropy (12) with k instead of n. Taking m, $\frac{1}{\epsilon}$, R, $k_0 \to \infty$ and $\omega \to 0$ in this estimate, one obtains

$$\chi\Big((x_j)_{1\leq j\leq k} : (p_i^{(v)})_{\substack{1\leq i\leq r_v\\ 1\leq v\leq f}}, (q_l^{(v)})_{\substack{1\leq l\leq s_v\\ 1\leq v\leq f}}, \{e_{jl}^{(v)}\}_{j,l,v}, \{f_{jl}^{(v)}\}_{j,l,v}, (z_u)_{1\leq u\leq p}\Big) < \chi(x_1, \ldots, x_k) \tag{13}$$

where $(p_i^{(v)})_{\substack{1\leq i\leq r_v\\ 1\leq v\leq f}}$, $(q_l^{(v)})_{\substack{1\leq l\leq s_v\\ 1\leq v\leq f}}$, $\{e_{jl}^{(v)}\}_{j,l,v}$, $\{f_{jl}^{(v)}\}_{j,l,v}$, $(z_u)_{1\leq u\leq p}$ are as in Proposition 3.4. If $\mathcal{A}_t$ denotes the von Neumann algebra $\{x_1, \ldots, x_t\}''$ and E_t the conditional expectation onto it, then

$$\Big((x_j)_{1\leq j\leq k}, (E_t(p_i^{(v)}))_{\substack{1\leq i\leq r_v\\ 1\leq v\leq f}}, (E_t(q_l^{(v)}))_{\substack{1\leq l\leq s_v\\ 1\leq v\leq f}}, \{E_t(e_{jl}^{(v)})\}_{j,l,v}, \{E_t(f_{jl}^{(v)})\}_{j,l,v}, (E_t(z_u))_{1\leq u\leq p}\Big)_{t\geq 1} \tag{14}$$

converges in distribution as $t \to \infty$ to

$$\Big((x_j)_{1\leq j\leq k}, (p_i^{(v)})_{\substack{1\leq i\leq r_v\\ 1\leq v\leq f}}, (q_l^{(v)})_{\substack{1\leq l\leq s_v\\ 1\leq v\leq f}}, \{e_{jl}^{(v)}\}_{j,l,v}, \{f_{jl}^{(v)}\}_{j,l,v}, (z_u)_{1\leq u\leq p}\Big)$$

therefore

$$\chi\Big((x_j)_{1\leq j\leq k} : (E_t(p_i^{(v)}))_{\substack{1\leq i\leq r_v\\ 1\leq v\leq f}}, (E_t(q_l^{(v)}))_{\substack{1\leq l\leq s_v\\ 1\leq v\leq f}}, \{E_t(e_{jl}^{(v)})\}_{j,l,v}, \{E_t(f_{jl}^{(v)})\}_{j,l,v}, (E_t(z_u))_{1\leq u\leq p}\Big) < \chi(x_1, \ldots, x_k) \tag{15}$$

for some large integer $t > k$. But this leads to a contradiction:

$$\chi(x_1,\dots,x_t) = \chi\Bigg((x_j)_{1\le j\le t} : (E_t(p_i^{(v)}))_{\substack{1\le i\le r_v\\ 1\le v\le f}}, (E_t(q_l^{(v)}))_{\substack{1\le l\le s_v\\ 1\le v\le f}}, \{E_t(e_{jl}^{(v)})\}_{j,l,v}, \{E_t(f_{jl}^{(v)})\}_{j,l,v}, (E_t(z_u))_{1\le u\le p}\Bigg) \le \chi\Bigg((x_j)_{1\le j\le k} : (E_t(p_i^{(v)}))_{\substack{1\le i\le r_v\\ 1\le v\le f}}, (E_t(q_l^{(v)}))_{\substack{1\le l\le s_v\\ 1\le v\le f}}, \{E_t(e_{jl}^{(v)})\}_{j,l,v}, \{E_t(f_{jl}^{(v)})\}_{j,l,v}, (E_t(z_u))_{1\le u\le p}\Bigg) + \chi(x_{k+1},\dots,x_t) < \chi(x_1,\dots,x_k) + \chi(x_{k+1},\dots,x_t) = \chi(x_1,\dots,x_t). \tag{16}$$

□

Corollary 3.6. *If $\mathcal{P} \subset \mathcal{L}(\mathbb{F}_n)$ is a subfactor of finite index and if $n \ge p + 2f + 2$, then $\mathcal{P}$ can not be asymptotically decomposed as*

$$\lim_{\omega\to 0}{}^{||\cdot||_2} \sum_{\substack{1\le j_1,\dots,j_{t+1}\le f\\ 1\le t\le d}} \mathcal{N}_{j_1}^\omega \mathcal{Z}^\omega \mathcal{N}_{j_2}^\omega \mathcal{Z}^\omega \dots \mathcal{N}_{j_t}^\omega \mathcal{Z}^\omega \mathcal{N}_{j_{t+1}}^\omega,$$

where $\{\mathcal{Z}^\omega\}_\omega$ contain p self-adjoint elements of $\mathcal{P}$, $\{\mathcal{N}_1^\omega,\dots,\mathcal{N}_f^\omega\}_\omega$ are nonprime subfactors of $\mathcal{P}$, and $d \ge 1$ is an integer.

Proof. Since $\mathcal{P} \subset \mathcal{L}(\mathbb{F}_n)$ is a subfactor of finite index, $\mathcal{L}(\mathbb{F}_n)$ can be obtained from $\mathcal{P}$ with the basic construction ([Jo], [JoSu]): there exists a subfactor $\mathcal{Q} \subset \mathcal{P}$ such that $\mathcal{L}(\mathbb{F}_n) =< \mathcal{P}, e_\mathcal{Q} >$, where $e_\mathcal{Q}$ is the Jones projection associated to the inclusion $\mathcal{Q} \subset \mathcal{P}$. But $< \mathcal{P}, e_\mathcal{Q} >= \mathcal{P}e_\mathcal{Q}\mathcal{P}$ ([JoSu]), hence $\mathcal{L}(\mathbb{F}_n)$ can be decomposed as $\mathcal{P}e_\mathcal{Q}\mathcal{P}$. Apply now Theorem 3.5. □

Corollary 3.7. *If $n \ge p+2f+1$, then the free group factor $\mathcal{L}(\mathbb{F}_n)$ can not be decomposed as*

$$\overline{\mathrm{sp}}^w \sum_{\substack{1\le j_1,\dots,j_{t+1}\le f\\ 1\le t\le d}} \mathcal{N}_{j_1}\mathcal{Z}\mathcal{N}_{j_2}\mathcal{Z}\dots\mathcal{N}_{j_t}\mathcal{Z}\mathcal{N}_{j_{t+1}},$$

where $\mathcal{Z} \subset \mathcal{L}(\mathbb{F}_n)$ contains p self-adjoint elements, $\mathcal{N}_1,\dots,\mathcal{N}_f$ are nonprime subfactors of $\mathcal{L}(\mathbb{F}_n)$, and $d \ge 1$ is an integer. Moreover, if $\mathcal{P} \subset \mathcal{L}(\mathbb{F}_n)$ is a subfactor of finite index and if $n \ge p + 2f + 2$, then $\mathcal{P}$ also can not be decomposed as

$$\overline{\mathrm{sp}}^w \sum_{\substack{1\le j_1,\dots,j_{t+1}\le f\\ 1\le t\le d}} \mathcal{N}_{j_1}\mathcal{Z}\mathcal{N}_{j_2}\mathcal{Z}\dots\mathcal{N}_{j_t}\mathcal{Z}\mathcal{N}_{j_{t+1}},$$

for any subset $\mathcal{Z}$ containing p self-adjoint elements of $\mathcal{P}$, any $\mathcal{N}_1,\dots,\mathcal{N}_f$ nonprime subfactors of $\mathcal{P}$, and any integer $d \ge 1$.

Proof. Follows from Theorem 3.5 and Corollary 3.6, for $\mathcal{Z}^\omega = \mathcal{Z}$, $\mathcal{N}_1^\omega = \mathcal{N}_1,\dots,\mathcal{N}_f^\omega = \mathcal{N}_f$. □

Corollary 3.8 settles a conjecture of L. Ge and S. Popa ([GePo]) in the case $n = \infty$. Recall that for a type II_1-factor $\mathcal{M}$ one defines $\ell_h(\mathcal{M}) = \min\{f \in \mathbb{N} \mid \exists$ hyperfinite $\mathcal{R}_1, \ldots, \mathcal{R}_f \subset M$ s.t. $\overline{\text{sp}}^w \mathcal{R}_1\mathcal{R}_2 \ldots \mathcal{R}_f = \mathcal{M}\}$. Note that the definition of hyperfinite dimension is given in terms of hyperfinite subalgebras. If one defined the hyperfinite dimension in terms of hyperfinite subfactors instead of hyperfinite subalgebras, then the proof of Corollary 3.8 would have followed immediately from Corollary 3.7. But with Definition 3.1, we need the asymptotic indecomposability result from Theorem 3.5.

Corollary 3.8. $\ell_h(\mathcal{L}(\mathbb{F}_n)) \geq [\frac{n-2}{2}] + 1 \;\forall 4 \leq n \leq \infty$.

Proof. If $\ell_h(\mathcal{L}(\mathbb{F}_n)) \leq [\frac{n-2}{2}]$, then $\mathcal{L}(\mathbb{F}_n) = \overline{\text{sp}}^w \mathcal{R}_1\mathcal{R}_2 \ldots \mathcal{R}_f$ for some hyperfinite subalgebras $\mathcal{R}_1, \ldots, \mathcal{R}_f$ and some integer f with $n \geq 2f + 2$. Let $m \geq 1$, $y_1, \ldots, y_m \in \mathcal{L}(\mathbb{F}_n)$ and $\omega > 0$ be fixed. Then there exist finite dimensional subalgebras $\mathcal{B}_v^\omega = \mathcal{B}_v(y_1, \ldots, y_m; \omega) \subset \mathcal{R}_v$, $1 \leq v \leq f$, such that

$$\text{dist}_{||\cdot||_2}\left(y_j, \mathcal{B}_1^\omega \mathcal{B}_2^\omega \ldots \mathcal{B}_f^\omega\right) < \omega \;\forall 1 \leq j \leq m.$$

Each finite dimensional subalgebra $\mathcal{B}_v^\omega$ is contained in a copy of the hyperfinite II_1-factor, say $\mathcal{B}_v^\omega \subset \mathcal{R}_v^\omega = \mathcal{R}_v^\omega(y_1, \ldots, y_m; \omega) \subset \mathcal{L}(\mathbb{F}_n)$. Consequently,

$$\text{dist}_{||\cdot||_2}\left(y_j, \mathcal{R}_1^\omega \mathcal{R}_2^\omega \ldots \mathcal{R}_f^\omega\right) < \omega \;\forall 1 \leq j \leq m,$$

hence $\mathcal{L}(\mathbb{F}_n)$ admits an asymptotic decomposition of the form

$$\lim_{\omega \to 0}{}^{||\cdot||_2} \mathcal{R}_1^\omega \mathcal{R}_2^\omega \ldots \mathcal{R}_f^\omega,$$

in contradiction with Theorem 3.5 as $\mathcal{R}_1^\omega, \ldots, \mathcal{R}_f^\omega$ are nonprime and $n \geq 2f + 2$. □

Corollary 3.9. *If $\mathcal{P} \subset \mathcal{L}(\mathbb{F}_n)$ is a subfactor of finite index and $5 \leq n \leq \infty$, then $\ell_h(\mathcal{P}) \geq [\frac{n-3}{2}] + 1$.*

Proof. Follows from Corollary 3.6. □

4. Indecomposability over abelian subalgebras

Another estimate of free entropy is used to prove that the free group factor $\mathcal{L}(\mathbb{F}_n)$ does not admit an asymptotic decomposition of the form

$$\lim_{\omega \to 0}{}^{||\cdot||_2} \sum_{\substack{1 \leq j_1, \ldots, j_{t+1} \leq f \\ 1 \leq t \leq d}} \mathcal{A}_{j_1}^\omega \mathcal{Z}^\omega \mathcal{A}_{j_2}^\omega \mathcal{Z}^\omega \ldots \mathcal{A}_{j_t}^\omega \mathcal{Z}^\omega \mathcal{A}_{j_{t+1}}^\omega,$$

where $\{\mathcal{A}_1^\omega, \ldots, \mathcal{A}_f^\omega\}$ are abelian subalgebras of $\mathcal{L}(\mathbb{F}_n)$, $\{\mathcal{Z}^\omega \subset \mathcal{L}(\mathbb{F}_n)\}_\omega$ are subsets with p self-adjoint elements, $d \geq 1$ is an arbitrary integer, and $n \geq p + 2f + 1$. Similarly, for free group subfactors one has the following: if $n \geq p + 2f + 2$ and $\mathcal{P} \subset \mathcal{L}(\mathbb{F}_n)$ is a subfactor of finite index, then $\mathcal{P}$ does not admit such an asymptotic decomposition either. In particular, the abelian dimension of $\mathcal{L}(\mathbb{F}_n)$ is $\geq [\frac{n-2}{2}] + 1$ and the abelian dimension of $\mathcal{P}$ is $\geq [\frac{n-3}{2}] + 1$. For $n = \infty$ this proves the second part of L. Ge's and S.

Popa's ([GePo]) conjecture: the abelian dimension of free group factors is infinite. The definitions of abelian dimension and asymptotic decomposition over abelian subalgebras are given next.

Definition 4.1. ([GePo]) If $\mathcal{M}$ is a II_1-factor, then the abelian dimension of $\mathcal{M}$, denoted $\ell_a(\mathcal{M})$, is defined as the smallest positive integer $f \in \mathbb{N}$ with the property that there exist abelian subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_f \subset \mathcal{M}$ such that $\overline{\mathrm{sp}}^w \mathcal{A}_1 \mathcal{A}_2 \ldots \mathcal{A}_f = \mathcal{M}$. If there is no such positive integer f, then by definition, $\ell_a(\mathcal{M}) = +\infty$.

Definition 4.2. A type II_1-factor $\mathcal{M}$ admits an asymptotic decomposition over abelian subalgebras, denoted

$$\lim_{\omega \to 0}{}^{||\cdot||_2} \sum_{\substack{1 \le j_1, \ldots, j_{t+1} \le f \\ 1 \le t \le d}} \mathcal{A}_{j_1}^{\omega} \mathcal{Z}^{\omega} \mathcal{A}_{j_2}^{\omega} \mathcal{Z}^{\omega} \ldots \mathcal{A}_{j_t}^{\omega} \mathcal{Z}^{\omega} \mathcal{A}_{j_{t+1}}^{\omega},$$

provided that $\forall n \ge 1$ $\forall x_1, \ldots, x_n \in \mathcal{M}$ $\forall \omega > 0$ $\exists \mathcal{A}_1^{\omega} = \mathcal{A}_1(x_1, \ldots, x_n; \omega), \ldots, \mathcal{A}_f^{\omega} = \mathcal{A}_f(x_1, \ldots, x_n; \omega)$ abelian $*$-subalgebras of $\mathcal{M}$ $\exists \mathcal{Z}^{\omega} = \mathcal{Z}(x_1, \ldots, x_n; \omega) \subset \mathcal{M}$ containing p self-adjoint elements, such that

$$\mathrm{dist}_{||\cdot||_2}\left(x_j, \sum_{\substack{1 \le j_1, \ldots, j_{t+1} \le f \\ 1 \le t \le d}} \mathcal{A}_{j_1}^{\omega} \mathcal{Z}^{\omega} \mathcal{A}_{j_2}^{\omega} \mathcal{Z}^{\omega} \ldots \mathcal{A}_{j_t}^{\omega} \mathcal{Z}^{\omega} \mathcal{A}_{j_{t+1}}^{\omega} \right) < \omega \ \forall 1 \le j \le n.$$

Proposition 4.3 gives an estimate of the free entropy of a (finite) system of generators of a II_1-factor $\mathcal{M}$ which can be asymptotically decomposed as

$$\lim_{\omega \to 0}{}^{||\cdot||_2} \sum_{\substack{1 \le j_1, \ldots, j_{t+1} \le f \\ 1 \le t \le d}} \mathcal{A}_{j_1}^{\omega} \mathcal{Z}^{\omega} \mathcal{A}_{j_2}^{\omega} \mathcal{Z}^{\omega} \ldots \mathcal{A}_{j_t}^{\omega} \mathcal{Z}^{\omega} \mathcal{A}_{j_{t+1}}^{\omega}.$$

As in the statement of Proposition 3.4, the approximations in the $||\cdot||_2$-norm (1) hold for every $\omega > 0$ if the II_1-factor can be decomposed as above.

Proposition 4.3. *Let $z_1, \ldots, z_p$ be self-adjoint elements of a II_1-factor $\mathcal{M}$ and let $(\mathcal{A}_v)_{1 \le v \le f}$ be a family of abelian subalgebras of $\mathcal{M}$. Let $x_1, \ldots, x_n$ be self-adjoint generators of $\mathcal{M}$ and assume that there exist projections $p_1^{(v)}, \ldots, p_{r_v}^{(v)} \in \mathcal{A}_v$ and complex noncommutative polynomials $(\phi_j)_{1 \le j \le n}$ of degrees $\le d$ (where $d \ge 1$ is fixed) in the variables $(z_u)_{1 \le u \le p}$ such that*

$$\left|\left| x_j - \phi_j\left((p_i^{(v)})_{\substack{1 \le i \le r_v \\ 1 \le v \le f}}, (z_u)_{1 \le u \le p} \right) \right|\right|_2 < \omega, \ j = 1, \ldots, n, \tag{1}$$

where $\omega \in (0, a]$ is a given positive number, and such that in all monomials of every ϕ_j the projections $p_i^{(v)}$ and $p_k^{(w)}$ are separated by some z_u if $v \ne w$. Then

$$\chi(x_1, \ldots, x_n) \le C(n, p, a, d, f) + (n - p - 2f) \log \omega, \tag{2}$$

where $a = \max\{||x_j||_2 + 1 | 1 \le j \le n\}$ and $C(n, p, a, d, f)$ is a constant that depends only on n, p, a, d, f.

Proof. As in the proof of Proposition 3.4 we can assume that $\phi_j = \phi_j^* \,\forall 1 \leq j \leq n$ and fix $R > 0$. Consider an arbitrary element

$$\left((B_j)_{1\leq j\leq n}, (P_i^{(v)})_{\substack{1\leq i\leq r_v \\ 1\leq v\leq f}}, (Z_u)_{1\leq u\leq p} \right)$$

of

$$\Gamma_R \left((x_j)_{1\leq j\leq n}, (p_i^{(v)})_{\substack{1\leq i\leq r_v \\ 1\leq v\leq f}}, (z_u)_{1\leq u\leq p}; m, k, \epsilon \right)$$

for some large integers m, k and small $\epsilon > 0$. Eventually after further restricting m, ϵ, we can find mutually orthogonal projections $Q_1^{(v)}, \ldots, Q_{r_v}^{(v)} \in \mathcal{M}_k^{sa}$ with $\mathrm{rank}(Q_i^{(v)}) = [\tau(p_i^{(v)})k]$ $\forall 1 \leq i \leq r_v$, such that

$$\left|\left| B_j - \phi_j \left((Q_i^{(v)})_{\substack{1\leq i\leq r_v \\ 1\leq v\leq f}}, (Z_u)_{1\leq u\leq p} \right) \right|\right|_2 < \omega \;\forall 1 \leq j \leq n\;.$$

If $S_1^{(v)}, \ldots, S_r^{(v)} \in \mathcal{M}_k^{sa}$ are fixed, mutually orthogonal projections with $\mathrm{rank}(S_i^{(v)}) = [\tau(p_i^{(v)})k]$ for every $1 \leq i \leq r_v$, then there exists a unitary $U^{(v)} \in \mathcal{U}(k)$ such that $Q_i^{(v)} = U^{(v)*} S_i U^{(v)}$ $\forall 1 \leq i \leq r_v$. The previous inequality becomes

$$\left|\left| B_j - \phi_j \left((S_i^{(v)})_{\substack{1\leq i\leq r_v \\ 1\leq v\leq f}}, (Z_u)_{1\leq u\leq p}, (\mathrm{Re}(U^{(v)}), \mathrm{Im}(U^{(v)}))_{1\leq v\leq f} \right) \right|\right|_2 < \omega,$$

and all the components of ϕ_j are polynomials of degrees $\leq 3d+2$ in the last $p+2f$ variables. Reasoning as in the last part of the proof of Proposition 3.4 we can easily obtain now the estimate $\chi(x_1, \ldots, x_n) \leq C(n, p, a, d, f) + (n - p - 2f)\log\omega$. □

4.1. **Abelian dimension of free group factors.**

Theorem 4.4. *If $n \geq p + 2f + 1$, then the free group factor $\mathcal{L}(\mathbb{F}_n)$ does not admit an asymptotic decomposition of the form*

$$\lim_{\omega\to 0}{}^{||\cdot||_2} \sum_{\substack{1\leq j_1,\ldots,j_{t+1}\leq f \\ 1\leq t\leq d}} \mathcal{A}_{j_1}^\omega \mathcal{Z}^\omega \mathcal{A}_{j_2}^\omega \mathcal{Z}^\omega \ldots \mathcal{A}_{j_t}^\omega \mathcal{Z}^\omega \mathcal{A}_{j_{t+1}}^\omega,$$

where each subset $\mathcal{Z}^\omega$ contains p self-adjoint elements, $\mathcal{A}_1^\omega, \ldots, \mathcal{A}_f^\omega \subset \mathcal{L}(\mathbb{F}_n)$ are abelian $*$-subalgebras and $d \geq 1$ is an integer.

Proof. Apply Proposition 4.3 in the same manner Proposition 3.4 was used in the proof of Theorem 3.5. □

Corollary 4.5. *If $\mathcal{P} \subset \mathcal{L}(\mathbb{F}_n)$ is a subfactor of finite index and if $n \geq p + 2f + 2$, then $\mathcal{P}$ can not be asymptotically decomposed as*

$$\lim_{\omega\to 0}{}^{||\cdot||_2} \sum_{\substack{1\leq j_1,\ldots,j_{t+1}\leq f \\ 1\leq t\leq d}} \mathcal{A}_{j_1}^\omega \mathcal{Z}^\omega \mathcal{A}_{j_2}^\omega \mathcal{Z}^\omega \ldots \mathcal{A}_{j_t}^\omega \mathcal{Z}^\omega \mathcal{A}_{j_{t+1}}^\omega,$$

where each subset $\mathcal{Z}^\omega$ contains p self-adjoint elements of $\mathcal{P}$, $\mathcal{A}_1^\omega, \ldots, \mathcal{A}_f^\omega \subset \mathcal{P}$ are abelian $*$-subalgebras, and $d \geq 1$ is an integer.

Proof. It is a direct consequence of Theorem 4.4 and of decomposition $\mathcal{L}(\mathbb{F}_n) = \mathcal{P}e_{\mathcal{Q}}\mathcal{P}$ (see the proof of Corollary 3.6). $\square$

Corollary 4.6. *If $n \geq p + 2f + 1$, then the free group factor $\mathcal{L}(\mathbb{F}_n)$ can not be decomposed as*

$$\overline{\mathrm{sp}}^{w} \sum_{\substack{1 \leq j_1, \ldots, j_{t+1} \leq f \\ 1 \leq t \leq d}} \mathcal{A}_{j_1}\mathcal{Z}\mathcal{A}_{j_2}\mathcal{Z}\ldots\mathcal{A}_{j_t}\mathcal{Z}\mathcal{A}_{j_{t+1}},$$

where $\mathcal{Z} \subset \mathcal{L}(\mathbb{F}_n)$ contains p self-adjoint elements, $\mathcal{A}_1, \ldots, \mathcal{A}_f$ are abelian $*$-subalgebras of $\mathcal{L}(\mathbb{F}_n)$, and $d \geq 1$ is an integer. Moreover, if $\mathcal{P} \subset \mathcal{L}(\mathbb{F}_n)$ is a subfactor of finite index and if $n \geq p + 2f + 2$, then $\mathcal{P}$ also can not be decomposed as

$$\overline{\mathrm{sp}}^{w} \sum_{\substack{1 \leq j_1, \ldots, j_{t+1} \leq f \\ 1 \leq t \leq d}} \mathcal{A}_{j_1}\mathcal{Z}\mathcal{A}_{j_2}\mathcal{Z}\ldots\mathcal{A}_{j_t}\mathcal{Z}\mathcal{A}_{j_{t+1}},$$

for any subset $\mathcal{Z}$ containing p self-adjoint elements of $\mathcal{P}$, any $\mathcal{A}_1, \ldots, \mathcal{A}_f$ abelian $*$-subalgebras of $\mathcal{P}$, and any integer $d \geq 1$.

Proof. Apply Theorem 4.4 and Corollary 4.5, for $\mathcal{Z}^{\omega} = \mathcal{Z}$, $\mathcal{A}_1^{\omega} = \mathcal{A}_1, \ldots,$ $\mathcal{A}_f^{\omega} = \mathcal{A}_f$. $\square$

Corollary 4.7 settles the second part of the conjecture of L. Ge and S. Popa ([GePo]), in the case $n = \infty$. As a reminder, $\ell_a(\mathcal{M})$ is defined as $\min\{f \in \mathbb{N} \mid \exists \mathcal{A}_1, \ldots, \mathcal{A}_f \subset \mathcal{M}$ abelian $*$-algebras s.t. $\overline{\mathrm{sp}}^{w}\mathcal{A}_1\mathcal{A}_2\ldots\mathcal{A}_f = \mathcal{M}\}$ for every type II_1-factor $\mathcal{M}$.

Corollary 4.7. $\ell_a(\mathcal{L}(\mathbb{F}_n)) \geq [\frac{n-2}{2}] + 1 \; \forall\, 4 \leq n \leq \infty.$

Proof. It follows from the first part of Corollary 4.6, for $\mathcal{Z} = \{1\}$. $\square$

Corollary 4.8. *If $\mathcal{P} \subset \mathcal{L}(\mathbb{F}_n)$ is a subfactor of finite index and $5 \leq n \leq \infty$, then $\ell_a(\mathcal{P}) \geq [\frac{n-3}{2}] + 1$.*

Proof. Apply the second part of Corollary 4.6. $\square$

Remark 4.9. One can combine both indecomposability properties of $\mathcal{L}(\mathbb{F}_n)$ into a single statement: if $n \geq p + 2f + 1$, then the free group factor $\mathcal{L}(\mathbb{F}_n)$ does not admit an asymptotic decomposition of the form

$$\lim_{\omega \to 0}{}^{\|\cdot\|_2} \sum_{\substack{1 \leq j_1, \ldots, j_{t+1} \leq f \\ 1 \leq t \leq d}} \mathcal{M}_{j_1}^{\omega}\mathcal{Z}^{\omega}\mathcal{M}_{j_2}^{\omega}\mathcal{Z}^{\omega}\ldots\mathcal{M}_{j_t}^{\omega}\mathcal{Z}^{\omega}\mathcal{M}_{j_{t+1}}^{\omega},$$

where each subset $\mathcal{Z}^{\omega}$ contains p self-adjoint elements, each $\mathcal{M}_1^{\omega}, \ldots, \mathcal{M}_f^{\omega} \subset \mathcal{L}(\mathbb{F}_n)$ is either a nonprime subfactor or an abelian $*$-subalgebra and $d \geq 1$ is an integer.

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